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Continuum Limit of $2D$ Spin Models with Continuous Symmetry and Conformal Quantum Field Theory

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Abstract

According to the standard classification of Conformal Quantum Field Theory (CQFT) in two dimensions, the massless continuum limit of the $O(2)$ model at the Kosterlitz-Thouless (KT) transition point should be given by the massless free scalar field; in particular the Noether current of the model should be proportional to (the dual of) the gradient of the massless free scalar field, reflecting a symmetry enhanced from $O(2)$ to $O(2) \times O(2)$. More generally, the massless continuum limit of a spin model with a symmetry given by a Lie group G should have an enhanced symmetry $G \times G$. We point out that the arguments leading to this conclusion contain two serious gaps: i) the possibility of 'non-trivial local cohomology' and ii) the possibility that the current is an ultralocal field. For the $2D$ $O(2)$ model we give analytic arguments which rule out the first possibility and use numerical methods to dispose of the second one. We conclude that the standard CQFT predictions appear to be borne out in the $O(2)$ model, but give an example where they would fail. We also point out that all our arguments apply equally well to any G symmetric spin model, provided it has a critical point at a finite temperature.

1.Introduction

Ever since the groundbreaking works of Belavin, Polyakov and Zamolodchikov (BPZ) [1] as well as Friedan, Qiu and Shenker (FQS) [2], it has been taken for granted that two-dimensional critical phenomena can be fully classified by the well known two-dimensional (rational) conformal quantum field theories (CQFTs). In theories with a continuous symmetry group G it is believed that the symmetry is 'doubled' to $G \times G$ [3] with left and right chiral theories both separately invariant under G. It is believed that essentially one only needs to construct the appropriate representation of the corresponding Kac-Moody (= current) algebra and out of it a representation of the Virasoro algebra by the so-called Sugawara construction, to be able to read off the properties of the critical theory.

Applying this philosophy to the model with the simplest continuous symmetry, namely the critical $O(2)$ model, Affleck concluded that the corresponding Kac-Moody and Virasoro algebras are those of the massless free field [3]. This means in particular that the Noether current is a gradient, hence its *curl* vanishes. But in Section 2 we point out a first gap in these conventional arguments which is related to the so-called problem of 'local cohomology' and we also provide a counterexample. In Appendix A we discuss the local cohomology problem in a little more detail. Our counterexample also shows the existence of critical theories that do not fit into the conformal classification. It is discussed in detail in Appendix B.

In Section 3 we give analytic arguments that show that in the case of the $O(2)$ model the situation is different from that in the counterexample and the *curl* of the current indeed vanishes in the continuum limit. These arguments make crucial use of the property of reflection positivity (RP). We concentrate on the $O(2)$ model as a typical example, but it should not be overlooked that our general aruments apply equally well to any $2D$ spin model with a continuous symmetry described by a Lie group G, provided it has a critical point at a finite value of the inverse temperature β.

In Section 4 we turn to another possible failure of the conformal classification: it could happen that in the continuum limit the current becomes 'ultralocal', i.e. its euclidean correlation functions are pure contact terms and the Minkowski space correlations vanish. To exclude this possibility we use numerical simulations as well as heuristic arguments. These Monte-Carlo simulations of the $O(2)$ model at its Kosterlitz-Thouless (KT) transition point also illustrate the features derived analytically in Section 3.

While thus, in the end, we confirm the conventional picture, we think it is important to realize that without the additional information provided here, there was no justification for accepting it.

2. Gaps in the Standard Arguments and a Counterexample.

Conventionally the arguments leading to the 'doubling' of the symmetry in the continuum limit of a critical theory and the splitting of the theory into two independent 'chiral' theories are given in the framework of Quantum Field Theory in Minkowski space [3]. Here we want to rephrase these arguments in the Euclidean setting, point out that one of the assumptions needed is not necessarily true and give an example violating that assumption.

Assume that we have a scale invariant continuum theory with a conserved current $j_\mu(x)$. Euclidean covariance requires that the two-point function $G_{\mu\nu}$ of j_μ is of the form

$$G_{\mu\nu} \equiv \langle j_\mu(0) j_\nu(x)\rangle = \delta_{\mu\nu}\frac{b}{x^2} + \frac{a x_\mu x_\nu}{(x^2)^2} \qquad (x \neq 0) \tag{1}$$

Imposing current conservation means

$$\partial_\mu G_{\mu\nu} = 0 \tag{2}$$

for $x \neq 0$, which implies

$$a = -2b \tag{3}$$

$$G_{\mu\nu}(x) = b\left(\frac{1}{x^2} - \frac{2x_\mu x_\nu}{(x^2)^2}\right) \tag{4}$$

This is, up to the factor b, equal to the two point function of $\partial_\mu\phi$ where ϕ is the massless free scalar field (it is irrelevant here that the massless scalar field does not exist as a Wightman field). If we look at the two-point function of the dual current $\epsilon_{\mu\nu} j_\nu$, it turns out to be

$$\tilde{G}_{\mu\nu} \equiv \epsilon_{\mu\lambda}\epsilon_{\rho\nu} G_{\lambda\rho} = G_{\mu\nu} \tag{5}$$

so the dual current two point function satisfies automatically the conservation law. Conservation of the two currents j and $\tilde{j}$ is equivalent to conservation of the two chiral currents $j_\pm = j_0 \pm j_1$ in Minkowski space.

By general properties of local quantum field theory (Reeh-Schlieder theorem, see [4]) it follows that the dual current is conserved as a quantum field. So the two conservation laws together imply that

$$j_\mu = \sqrt{b}\,\partial_\mu\phi, \tag{6}$$

where ϕ is the massless scalar free field, and also that

$$j_\mu = \sqrt{b}\,\epsilon_{\mu\nu}\partial_\nu\psi, \tag{7}$$

where ψ is another 'copy' of the massless scalar free field.

As presented, this argument is certainly correct. But it depends on the assumption that the Noether currents *exist* as Wightman fields, and this assumption is in fact

nontrivial and could a priori fail in the critical $O(2)$ model. A simple example of a Quantum Field Theory with a continuous symmetry in which the Noether current does not exist as a Wightman field is given by the two-component free field in $2D$ in the massless limit. It is simply given by a pair of independent Gaussian fields $\Phi^{(1)}, \Phi^{(2)}$, both with covariance

$$C(x) = \frac{1}{(2\pi)^2} \int d^2p \frac{e^{ipx}}{p^2 + m^2}. \tag{8}$$

where we are interested in the limit $m \to 0$. This system has a global $O(2)$ invariance rotating the two fields into each other. It is well known that the massless limit only makes sense for functions of the gradients of the fields. But the Noether current of the $O(2)$ symmetry is given by

$$j_\mu(x) = \Phi^{(1)}(x)\partial_\mu \Phi^{(2)}(x) - \Phi^{(2)}(x)\partial_\mu \Phi^{(1)}(x), \tag{9}$$

and it cannot be written as a function of the gradients. It is also easy to see directly that its correlation functions do not have a limit as $m \to 0$ (see Appendix B). The Noether current itself makes sense as a quantum field only if it is smeared with test functions f_μ satisfying

$$\int d^2x f_\mu(x) = 0 \tag{10}$$

On the other hand, it is not hard to see that $\phi_c(x) = curl(j)$ can be written as a function of the gradients:

$$\phi_c(x) = 2((\partial_2 \Phi^{(1)}(x))(\partial_1 \Phi^{(2)}(x)) - (\partial_1 \Phi^{(1)}(x))(\partial_2 \Phi^{(2)}(x))) \tag{11}$$

and its two-point function is of the form

$$\langle \phi_c(0)\phi_c(x)\rangle \propto \frac{1}{(x^2)^2} \tag{12}$$

In Appendix A we give some explicit formulae concerning this model starting from its lattice version.

The problem in the $O(2)$ model is then the following: it is conceivable that both *curl j* and *div j* have bona fide continuum limits, but the current itself does not. This is a so-called 'local cohomology' problem, that also arises in other contexts. We give a short discussion of this in Appendix B. In the next section we will use general arguments such as Reflection Positivity (RP) together with the fact that the $O(2)$ model becomes critical at a finite value of the inverse temperature β to rule out this possibility for the $O(2)$ model. Our arguments will show that both *curl j* and *div j* have correlations that are pure contact terms in the continuum limit; this means that in Minkowski space both the current and its dual are conserved, in accordance with Affleck's claim.

This leaves, however, still another possibility open, which would make the conformal classification inapplicable in the critical $O(2)$ model: it could happen that the current itself has correlations that are pure contact terms, in which case the Minkowski space Noether current would simply vanish in the continuum limit. We do not see any way to rule out this possibility by pure thought; but our numerical simulations reported in Section 4 make it very likely that this does not happen and the current is indeed a nonvanishing multiple of the gradient of a massless free field, as Affleck claims.

While our arguments establishing the enhancement of the symmetry to $G \times G$ would apply to any other $2D$ model with continuous symmetry group G possessing a massless phase at finite β, one would have to appeal to numerics to decide whether the current becomes ultralocal or not. The standard wisdom is that $2D$ $O(N)$ models with $N > 2$ have $\beta_{crt} = \infty$ ([5]). We disputed this scenario and argued that for any finite N, $\beta_{crt} < \infty$ [6, 7]. One may wonder what the numerics reveal; we give some preliminary report at the end of Section 4. It suggests that the situation for $O(N)$, $N > 2$ is not different from that for $O(2)$.

3. The Noether Current of the $O(2)$ Model: Analytic Arguments

The $O(2)$ model is determined by its standard Hamiltonian (action)

$$H = -\sum_{\langle ij \rangle} s(i) \cdot s(j) \tag{13}$$

where the sum is over nearest neighbor pairs on a square lattice and the spins $s(.)$ are unit vectors in the plane $\mathbb{R}^2$. As usual Gibbs states are defined by using the Boltzmann factor $\exp(-\beta H)$ together with the standard a priori measure on the spins first in a finite volume, and then taking the thermodynamic limit.

The model has been studied extensively both theoretically [5] and by Monte Carlo simulations (see for instance [8, 9, 10]). Its most interesting property is its so-called KT transition, named after Kosterliz and Thouless, from a high temperature phase with exponential clustering to a low temperature one with only algebraic decay of correlations; according to a recent estimate this transition takes place at $\beta_{KT} \approx 1.1197$ [10].

The nature of the transition is supposed to be peculiar, with exponential instead of the usual power-like singularities, but this is not our concern here. Instead we want to study the model at its transition point. We are in particular interested in the correlations of the Noether current, given by

$$j_\mu(i) = \beta\Big(s_1(i)s_2(i+\hat{\mu}) - s_2(i)s_1(i+\hat{\mu})\Big) = \beta \sin\Big(\phi(i+\hat{\mu}) - \phi(i)\Big) \tag{14}$$

where

$$s_1(i) = \cos(\phi(i)), s_2(i) = \sin(\phi(i)) \tag{15}$$

To our knowledge this observable has not been studied in the literature.

On a torus the current can be decomposed into 3 pieces, a longitudinal one, a transverse one and a constant (harmonic) piece. This decomposition is easiest in momentum space, and effected by the projections

$$P^T_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{(e^{ip_\mu}-1)(e^{-ip_\nu}-1)}{\sum_\alpha(2-2\cos p_\alpha)}\right)(1-\delta_{p0}), \tag{16}$$

$$P^L_{\mu\nu=} = \frac{(e^{ip_\mu}-1)(e^{-ip_\nu}-1)}{\sum_\alpha(2-2\cos p_\alpha)}(1-\delta_{p0}) \tag{17}$$

and

$$P^h_{\mu\nu} = \delta_{\mu\nu}\delta_{p0}. \tag{18}$$

with $p_\mu = 2\pi n_\mu/L$, $n_\mu = 0, 1, 2, ..., L-1$.

In the following we will mostly discuss these correlations in momentum space. In particular we study the tranverse momentum space 2-point function

$$\hat{F}^T(p, L) \equiv \hat{G}(0, p; L) = \langle|\hat{j}_1(0, p)|^2\rangle \tag{19}$$

(for $p \neq 0$; the hat denotes the Fourier transform)
and the longitudinal two-point function

$$\hat{F}^L(p, L) \equiv \hat{G}(p, 0; L) = \langle|\hat{j}_1(p, 0)|^2\rangle \tag{20}$$

(for $p \neq 0$).

Because the current is conserved, its divergence in the Euclidean world should be a pure contact term, and for dimensional reasons the two-point function should be proportional to a δ function, i.e.

$$\hat{F}^L(p, L) = const. \tag{21}$$

The constant is in fact determined by a Ward identity in terms of $E = \langle s(0)\cdot s(\hat{\mu})\rangle$: consider (for a suitable finite volume) the partition function

$$Z = \int \prod_i d\phi(i) \prod_{\langle ij\rangle} \exp(\beta\cos(\phi(i) - \phi(j)) \tag{22}$$

Replacing under the integral $\phi(i)$ by $\phi(i) + \alpha(i)$ does not change its value. So expanding in powers of α, all terms except the one of order α^0 vanish indentically in

$\alpha(i)$. This leads in a well-known fashion to Ward identities expressing the conservation of the current. Looking specifically at the second order term in α and Fourier transforming, we obtain

$$\langle |j_1(p,0)|^2 \rangle = \hat{F}^L(p,L) = \beta E \tag{23}$$

This is confirmed impressively by the Monte Carlo simulations which are reported in the next section.

The thermodynamic limit is obtained by sending $L \to \infty$ for fixed $p = 2\pi n/L$, so that in the limit p becomes a continuous variable ranging over the interval $[-\pi, \pi)$. The $O(2)$ model not only does not show spontaneous symmetry breaking according to the Mermin-Wagner theorem, but it has a unique infinite volume limit, as shown long ago by Bricmont, Fontaine and Landau [11]. In the next section we illustrate the convergence to the thermodynamic limit with Monte-Carlo simulations.

The continuum limit in a box, on the other hand, is obtained as follows: we take a fraction $rL \equiv L/l$ of L as the standard of length (since the system does not produce an intrinsic scale) and look at the correlations of $j^{ren}_\mu(x) = \frac{L}{l} j_\mu(i)$ with $x = \frac{il}{L}$ for $L \to \infty$; l becomes the size of the box in 'physical' units (see [12] for the principles of this construction). In Fourier space that means that one has to study e.g. the behavior of $\hat{F}^T(p;L)$ for fixed n where $p = 2\pi n/L$. We will prove that this limit is trivial: it is independent of p, corresponding to a contact term in x-space. This behavior is also illustrated by our numerical simulations in the next section.

More precisely we want to prove rigorously that the continuum limit of the thermodynamic limits $\hat{F}^T(p,\infty)$ and $\hat{F}^L(p,\infty)$ of $\hat{F}^T(p,L)$ and $\hat{F}^L(p,L)$ are constants; the second fact is of course again just a restatement of the Ward identity (12), whereas the first one expresses the vanishing of *curl* j in the continuum, thus confirming Affleck's claim regarding the enhancement of the continuous symmetry.

The continuum limit in the infinite volume is obtained as follows: let $\hat{F}(p;\infty) \equiv \hat{T}(p)$ be the Fourier transform of the one-dimensional lattice function $T(n)$. In general $\hat{T}$ has to be considered as a distribution on $[-\pi, \pi)$, and it can be extended to a periodic distribution on the whole real line. The continuuum limit of $T(n)$ also has to be considered in the sense of distributions; it is obtained by introducing an integer N as the unit of length, making the lattice spacing equal to $1/N$. For an arbitrary test function f (infinitely differentiable and of compact support) on the real axis we then have to consider the limit $N \to \infty$ of

$$(T,f)_N \equiv \sum_n f(\frac{n}{N}) T(n). \tag{24}$$

We claim that the right hand side of this is equal to

$$\frac{1}{2\pi} \int_\infty^\infty dq \hat{T}(\frac{q}{N}) \hat{f}(q). \tag{25}$$

Proof: Insert in eq.(24)

$$T(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dp\hat{T}(p)e^{ipn} \tag{26}$$

and

$$f(\frac{n}{N}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dp\hat{f}(p)e^{ipn/N} \tag{27}$$

and use the identity

$$\sum_n e^{ipn+iqna} = 2\pi\sum_r \delta(p+qa+2\pi r) \tag{28}$$

This produces, after carrying out the trivial integral over q using the δ distribution,

$$\frac{N}{2\pi}\int_{-\pi}^{\pi} dp \sum_{r=-\infty}^{\infty} \hat{T}(-p)\hat{f}((p+2\pi r)N) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty}\int_{-N\pi}^{N\pi} dq\hat{T}(-\frac{q}{N})\hat{f}(q+2\pi Nr) \tag{29}$$

Finally, using the periodic extension of $\hat{T}(p)$, this becomes what is claimed in eq.(25).

From eq.(25) one sees that what is relevant for the continuum limit is the small momentum behavior of $\hat{T}(p)$. In particular, if $lim_{p\to 0}\hat{T}(p) \equiv \hat{T}(0)$ exists, we obtain

$$\lim_{N\to\infty}(T,f)_N = \frac{1}{2\pi}\hat{T}(0)\int dq\hat{f}(q) = \frac{1}{2\pi}f(0)\hat{T}(0) \tag{30}$$

expressing the fact that in this case the limit of T is a pure contact term.

Next we use reflection positivity (RP) of the Gibbs measure formed with the standard action (see for instance [13]) on the periodic lattice. Reflection positivity means that expectation values of the form

$$\langle A\theta(A)\rangle, \tag{31}$$

are nonnegative, where A is an observable depending on the spins in the 'upper half' of the lattice ($\{x|x_1>0\}$, and $\theta(A)$ is the complex conjugate of the same function of the spins at the sites with x_1 replaced by $-x_1$. Applied to the current two-point functions this yields:

$$F^L(x_1,L) = \sum_{x_2}\langle j_1(x_1,x_2)j_1(0,0)\rangle \le 0 \tag{32}$$

for $x_1 \neq 0$ and

$$F^T(x_1,L) = \sum_{x_2}\langle j_2(x_1,x_2)j_2(0,0)\rangle \ge 0 \tag{33}$$

for all x_1. From these two equations it follows directly that

$$0 \le \hat{F}^T(p,L) \le \hat{F}^T(0,L) = \hat{F}^L(0,L) \le \hat{F}^L(p,L) = \beta E \tag{34}$$

These inequalities remain of course true in the thermodynamic limit, but we have to be careful with the order of the limits. If we define $\hat{F}^T(p,\infty)$ and $\hat{F}^L(p,\infty)$ as the Fourier transforms of $\lim_{L\to\infty} F^T(x,L)$ and $\lim_{L\to\infty} F^L(x,L)$, respectively, one conclusion can be drawn immediately:

Proposition: $\hat{F}^T(p,\infty)$ and $\hat{F}^L(p,\infty)$ are continuous functions of $p \in [-\pi,\pi)$.

The proof is straightforward, because due to the inequalities (32) (33) and (34) together with the finiteness of β_{KT} the limiting functions F^L and F^T in x-space are absolutely summable.

But it is not assured that the limits $L \to \infty$ and $p \to 0$ can be interchanged, nor that the thermodynamic limit and Fourier transformation can be interchanged. On the contrary, by the numerics presented in the next section, as well as finite size scaling arguments, it is suggested that

$$\lim_{p\to 0}\lim_{L\to\infty} \hat{F}^L(p,L) > \lim_{L\to\infty} \hat{F}^L(0,L) \tag{35}$$

and therefore also

$$\lim_{p\to 0}\lim_{L\to\infty} \hat{F}^L(p,L) > \lim_{p\to 0}\lim_{L\to\infty} \hat{F}^T(p,L). \tag{36}$$

This will play an important role in the justification of Affleck's claim. But for now we want to show only the following:

Proposition: In the continuum limit both $\hat{F}^L(p,\infty)$ and $\hat{F}^T(p,\infty)$ ($p \neq 0$) converge to constants.

Proof: The proof was essentially given above. in eq.(26) to eq.(30). We only have to notice that due to eq.(34) $\lim_{p\to 0} \hat{F}^L(p,\infty)$ and $\lim_{p\to 0} \hat{F}^T(p,\infty)$ exist.

In spite of this result, Affleck's claim could still fail in a different way if $\hat{F}^T(p,\infty)$ and $\hat{F}^L(p,\infty)$ converged to the same constant in the continuum limit. Let us denote the continuum limit of $\hat{F}^T(p,\infty)$ by g. Then the current-current correlation in this limit is

$$\hat{\langle j_\mu j_\nu \rangle}(p) = \beta E P^L_{\mu\nu} + g P^T_{\mu\nu} = g\delta_{\mu\nu} + (\beta E - g)\frac{p_\mu p_\nu}{p^2}. \tag{37}$$

So we see that if $g = \beta E$, the current-current correlation reduces to a pure contact term and vanishes in Minkowski space. Above we proved only that

$$g \leq \beta E \tag{38}$$

In the next section we will invoke numerical simulation data together with finite size scaling arguments to rule out this possibility and finally justify Affleck's claim for $O(2)$.

4. The Noether Current: Numerical Simulations

As remarked before, a recent estimate for the transition point is [10]

$$\beta_{KT} = 1.1197 \tag{39}$$

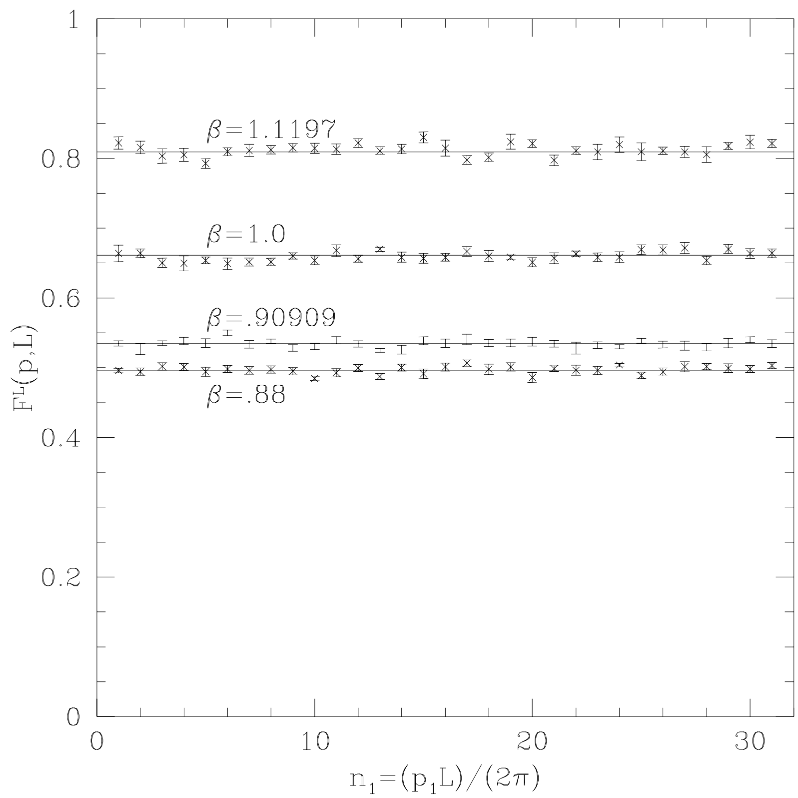

Figure 1: The longitudinal current correlation

Of course this number is not exact, but for our purpose it is sufficient that the correlation length is so large that on the lattices we can simulate it may be treated as infinite.

In Fig.1 we report some data of the longitudinal current-current correlation $\hat{F}^L(p, L)$, taken on a 64×64 lattice at different values of β. The figure illustrates how well the Ward identity eq.(23) is satisfied by our data.

For the transverse current-current correlation $\hat{F}^T(p, L)$ we took data at $\beta = 1.1197$ on lattices of linear extent $L = 50$, 100, 200 and 400. For the three smaller L values we took three or four runs of 500,000 clusters each, whereas for $L = 400$ we only have one such run. The thermodynamic limit is obtained by sending $L \to \infty$ for fixed

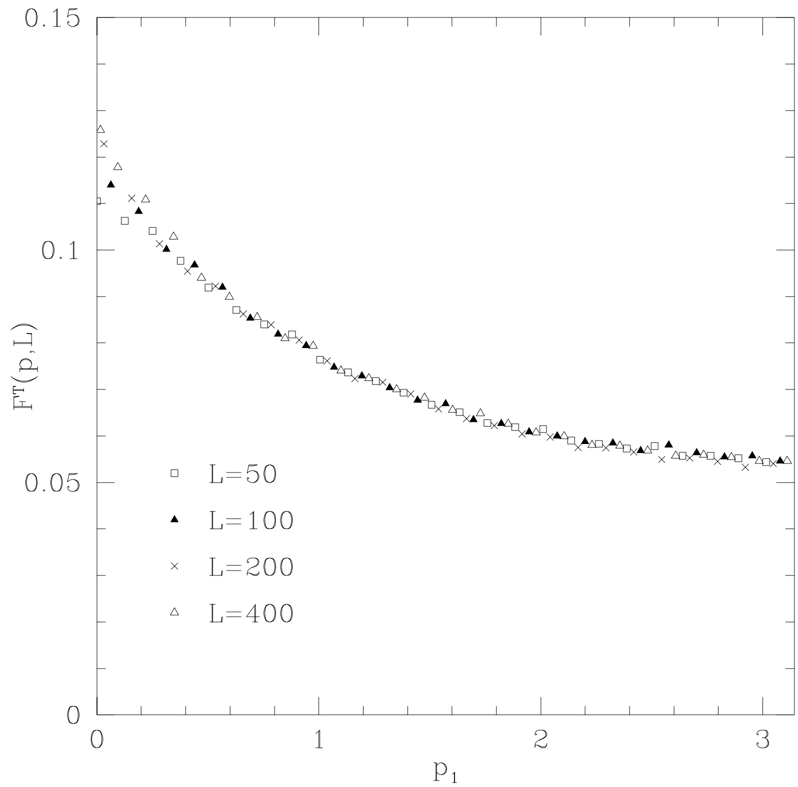

Figure 2: Transverse current correlation: thermodynamic limit

$p_1 = 2\pi n_1/L$. In Fig. 2 we show the values of $\hat{F}^T(p, L)$ plotted against p for different values of L. The figure illustrates the convergence towards the thermodynamic limit, although there might be some nonuniformity for $p \to 0$.

In Fig. 3 we plot $F^T(0, L) - F^T(p, L)$ for $L = 50$, 100, 200, and 400 against the continuum momentum paramter $n = pL/2\pi$. This figure illustrates how this difference converges to zero as we approach the continuum limit, in accordance with the analytic proof given in the previous section.

Let us finally turn to the question left open in the previous section, namely whether the continuum limit g of $\hat{F}^T(p, \infty)$ is equal to βE or not. For $\beta < \beta_{KT}$ the current two point function is decaying exponentially, hence its Fourier transform is continuous (and even real analytic). The same applies then to the longitudinal and transverse parts $\hat{F}^L(p, \infty)$ and $\hat{F}^T(p, \infty)$; in particular

$$\hat{F}^T(0, \infty) = \hat{F}^L(0, \infty) = \beta E \tag{40}$$

by the Ward identity eq.(23).

That does not, however, imply that at $\beta = \beta_{KT}$ $\lim_{p \to 0} \hat{F}^T(p, \infty) = \beta E$, because

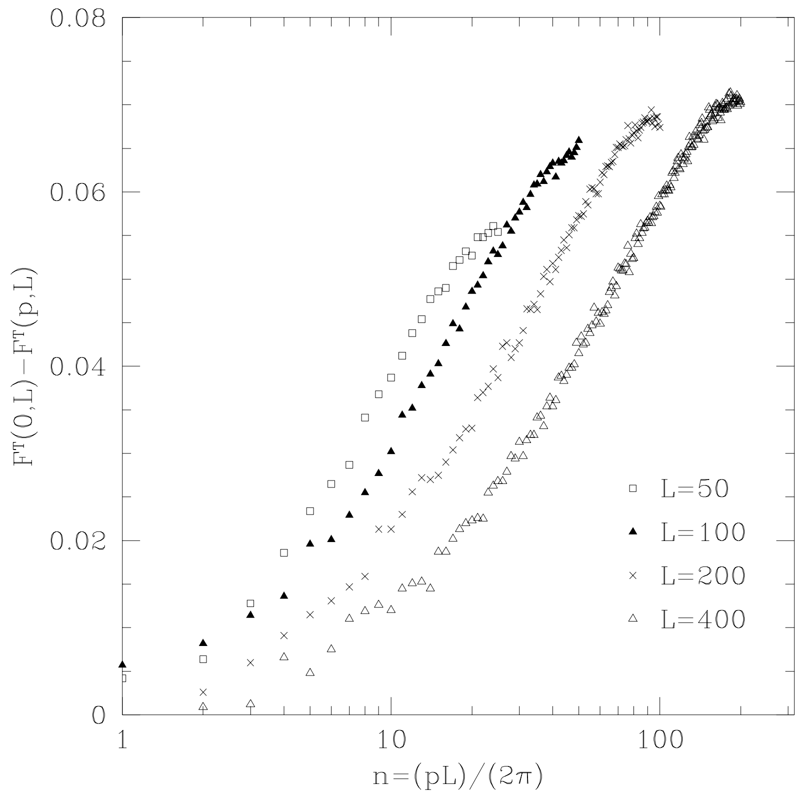

Figure 3: The transverse current correlation on different lattices

the current two-point function cannot be expected to be absolutely summable there. On the contrary, if we can find that

$$\lim_{L\to\infty} \hat{F}^L(0,L) < \beta E \tag{41}$$

this implies also

$$g = \lim_{p\to 0} \hat{F}^T(p,\infty) < \beta E \tag{42}$$

because by eq.(34) $\hat{F}^T(p,\infty) \leq \lim_{L\to\infty} \hat{F}^L(0,L)$.

The MC data taken at β_{KT} and listed in Tab.1 indicate that

$$d \equiv \beta E - \hat{F}^L(0,L) = \hat{F}^L(\frac{2\pi}{L},L) - \hat{F}^L(0,L) \tag{43}$$

goes to a positive number ($< .68$ but probably $> .6$), suggesting that indeed $g < \beta E$. But the question is whether this 'discontinuity' is a finite volume artefact or not. To address this issue we took data at $\beta < \beta_{KT}$ keeping the ratio L/ξ fixed while increasing ξ. In this approach the massless continuum limit would correspond to

$L/\xi \to 0$ (while $L/\xi \to \infty$ would correspond to the massive continuum limit in a thermodynamic box). Actually we use L/ξ_{eff} instead of L/ξ as an independent variable, where ξ_{eff} is the effective correlation length measured on the lattice of size L; in the finite size scaling limit the two variables are equivalent, because L/ξ_{eff} becomes a unique monotonic function of L/ξ. The data listed in Tab.2 indicate that $d(L) = \beta E - \hat{F}^L(p, L)$ (the 'discontinuity' of $\hat{F}^L(p, L)$ at $p = 0$) depends only on L/ξ_{eff} in agreement with finite size scaling, and that it goes to a value above .6 in the massless continuum limit which is reached around $L/\xi_{eff} \approx 1.3$. Together with the data taken at β_{KT}, this tells us that $\lim_{L\to\infty} d(L)$ is somewhere between .6 and .68 (it actually might be equal to $2/\pi$). The two data sets together, in any case, provide convincing evidence that the Noether current is not an ultralocal field.

In closing we want to repeat that none of our analytic considerations in this paper were specific to the $O(2)$ model: they apply equally well to the $O(N)$ model for any N, provided it has (as we believe) a second order phase transition at some finite value β_{crt}. We ran some exploratory tests in the $O(3)$ model. If we run at $\beta = 3$, a value at which this model may be in its massless phase, unfortunately any lattice amenable to numerical simulation is so 'frozen' that Monte Carlo data simply reproduce perturbation theory. Thus the only alternative is to run in the massive phase, with β and L chosen such that we see massless behavior ($L < \xi$) yet L large enough so that the model can exhibit nonperturbative behavior. Our data revealed a behavior similar to the one found in $O(2)$: at fixed L/ξ there is a 'discontinuity' which scales with increasing L and is a function of L/ξ only. However that should happen whether β_{crt} is finite or not, hence all we can say is that if β_{crt} is finite, the situation is quite similar to the one encountered in the $O(2)$ model.

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Appendix A: The current of the two-component free field

We first consider the massive two-component scalar field in a finite periodic box of size L on the unit lattice. It consists of two independent Gaussian lattice fields Φ_1 and Φ_2, both with covariance

$$C(x-y) = \frac{1}{L^2} \sum_{n_1,n_2=0}^{L-1} \frac{\exp(2\pi i n \cdot (x-y)/L)}{m^2 + \sum_\mu (2 - 2\cos(2\pi n_\mu / L)}. \tag{44}$$

The Noether current is given by an expression analogous to eq.(3), namely

$$j_\mu(x) = \Phi_1(x)\Phi_2(x+\hat{\mu}) - \Phi_2(x)\Phi_1(x+\hat{\mu}) \tag{45}$$

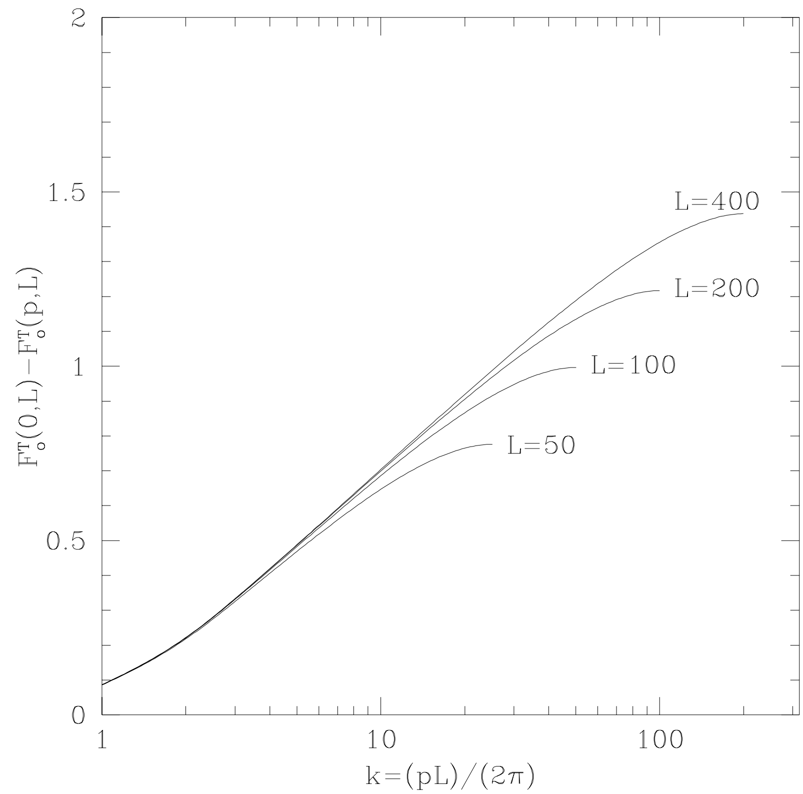

Figure 4: Transverse free current correlation: continuum limit

It is straightforward to compute the *curl* and the divergence of this current:

$$\begin{aligned}
curl\ j(x) &= j_1(x) - j_1(x+\hat{2}) + j_2(x+\hat{1})) - j_2(x) \\
&= \Big(\Phi_1(x) - \Phi_1(x+\hat{1}+\hat{2})\Big)\Big(\Phi_2(x+\hat{1}) - \Phi_2(x+\hat{2})\Big) \\
&- \Big(\Phi_2(x) - \Phi_2(x+\hat{1}+\hat{2})\Big)\Big(\Phi_1(x+\hat{1}) - \Phi_1(x+\hat{2})\Big)
\end{aligned} \tag{46}$$

$$div\ j(x) = \Phi_1(x)\Big(\Delta\Phi_2\Big)(x) - \Phi_2(x)\Big(\Delta\Phi_1\Big)(x). \tag{47}$$

It is obvious from these formulae that the massless limit of *curl* j exists, because it depends only on differences of Φ's, whereas for *div* j it does not. Likewise the two-point function of the transverse part of the current has a limit as $m \to 0$, whereas the two-point function of the longitudinal part does not.

Next we want to give explicit expressions for the two-point functions of the current in momentum space. We give separately the transverse and the longitudinal parts $F_o^T(p,L)$ and $F_o^L(p,L)$, respectively, choosing the momentum $p = (p_1, 0) =$

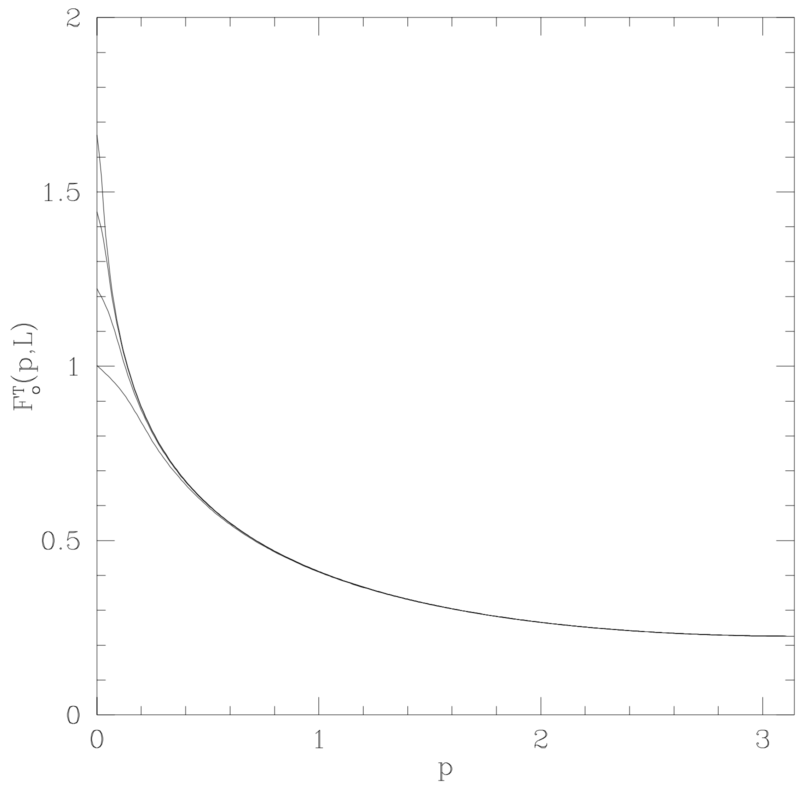

Figure 5: Transverse free current correlation: thermodynamic limit

$(2\pi k_1/L, 0)$ in the 1 direction:

$$
\begin{aligned}
F_o^T(p, L) = \quad & \tfrac{2}{L^2} \quad \sum_{n_1, n_2} \frac{1}{m^2 + 4 - 2\cos\frac{2\pi n_1}{L} - 2\cos\frac{2\pi n_2}{L}} \\
& \times \quad \frac{\left(1 - \cos\frac{2\pi n_2}{L}\right)}{m^2 + 4 - 2\cos\frac{2\pi(k_1 - n_1)}{L} - 2\cos\frac{2\pi n_2}{L}}
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
F_o^L(p, L) = \quad & \tfrac{2}{L^2} \quad \sum_{n_1, n_2} \frac{1}{m^2 + 4 - 2\cos\frac{2\pi n_1}{L} - 2\cos\frac{2\pi n_2}{L}} \\
& \times \quad \frac{\left(1 - \cos\frac{2\pi(2n_1 - k_1)}{L}\right)}{m^2 + 4 - 2\cos\frac{2\pi(k_1 - n_1)}{L} - 2\cos\frac{2\pi n_2}{L}}
\end{aligned}
\tag{49}
$$

The continuum limit of this function would be obtained by sending $L \to \infty$, keeping $k_1 = pL/2\pi$ fixed. It does not exist, but if we replace $F^T(p, L)$ by $F^T(0, L) - F^T(p, L)$, the limit does exist. This is illustrated in Fig. 4.

The thermodynamic limit, on the other hand, is obtained by sending $L \to \infty$, keeping $p = 2\pi k_1/L$ fixed. This limit does exist for $p \neq 0$, as illustrated in Fig. 5.

Appendix B: Local Cohomology

It has been noted long ago [15, 16, 17] that the imposition of locality (local commutativity, Einstain causality) may make the cohomology of Minkowski space nontrivial.

The problem of local cohomolgy may be stated as follows: assume that an antisymmetric tensor field $\Phi_{\mu_1,\ldots,\mu_k}(x)$ is given, which satisfies Wightman's axioms and is closed, i.e. satisfies

$$d\Phi \equiv d(\sum \Phi_{\mu_1,\ldots,\mu_k} dx^{\mu_1} \ldots d\mu_k = 0 \tag{50}$$

(in the notation of alternating differential forms).

The question is then under which conditions the field Φ is exact, i.e. there exists a local antisymmetric tensor field Ψ such that $\Phi = d\Psi$.

There are some well-known examples where the answer is 'no', even though Minkowski space is topologically trivial:

(1) the free Maxwell field F in dimension $D \geq 2$ [15];

(2) the gradient of the massless free scalar field ϕ in $2D$, because the field ϕ does not exist as a local (Wightman) field.

In this paper we came across a new $2D$ example: let

$$\Phi = \phi_c dx^1 dx^2 \tag{51}$$

where ϕ_c has the Euclidean two-point function

$$\langle \phi_c(0)\phi_c(x) \rangle = \frac{1}{(x^2)^2}. \tag{52}$$

Then Φ is trivially closed in $2D$, but it is not exact, i.e. there is no local vector field j_μ such that

$$\phi_c = \epsilon_{\mu\nu}\partial_\mu j_\nu \tag{53}$$

This example can be made more explicit by requiring ϕ_c to be a generalized free, i.e. Gaussian field, with its two-point function given by eq. (39). If we solve the differential equations that the two-point function of j_μ has to fulfill in order to satisfy eq.(40) and impose euclidean covariance, we find that there is no scale invariant solution. The covariant solutions are

$$G_{\mu\nu}(x) = -\delta_{\mu\nu}\frac{\ln x^2 + \lambda}{8x^2} + x_\mu x_\nu \frac{\ln x^2 + 1 + \lambda}{4x^2} \tag{54}$$

This is not the two point function of a local vector field, continued to euclidean times: it violates the so-called reflection positivity [18], because the logarithm changes sign. For the same reason it is also not the two point function of a random field.

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Tab.1: *The 'discontinuity'* $d(L) = \beta E - g(L)$ *at* β_{KT} *for different values of* L.

L	L/ξ_{eff}	$g(L)$	$d(L)$
25	1.2575	.09769	.7117(10)
50	1.2657	.10666	.7027(18)
100	1.2667	.11638	.6930(11)
200	1.2722	.12357	.6858(08)
400	1.2839	.12944	.6799(16)

Tab.2: *The 'discontinuity'* $d(L) = \beta E - g(L)$ *at various values of* $\beta < \beta_{KT}$ *and* L.

β	L	L/ξ_{eff}	$d(L)$
.93	12	1.8319	.3646(22)
.93	24	2.4441	.1926(24)
.93	36	3.2259	.0875(34)
.96	18	1.8314	.3636(24)
.96	36	2.4242	.1929(31)
.96	54	3.1996	.0926(34)
.99	32	1.8595	.3543(48)
.99	63	2.4537	.1919(64)
.99	64	2.4467	.1862(46)
.99	96	3.2110	.0909(43)
1.04	63	1.5868	.4709(59)
1.06	63	1.4549	.5411(49)
1.08	63	1.3722	.6012(36)
1.08	126	1.4188	.5818(46)
1.09	126	1.3990	.6174(18)